Any isometry between the spheres of 2-dimensional Banach spaces is linear

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Tingley's problem

We will deal with this question, that appeared in [8]:

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces and $\tau : S_X \to S_Y$ an onto isometry. Is τ the restriction of some linear isometry $\widetilde{\tau} : X \to Y$?

Mazur-Ulam Theorem

The first great result about the isometries in normed spaces gives a positive answer to the same Problem when the onto isometry is defined on the whole space (see [7]). Namely:

Mazur-Ulam Theorem (1932): Every onto isometry $\widetilde{\tau} : X \to Y$ between normed spaces is affine. So, $\widetilde{\tau}$ is linear whenever $\widetilde{\tau}(0) = 0$.

Mankiewicz's Theorem

A huge advance about isometries between normed spaces was Mankiewicz's Theorem, and it is, to the best of our knowledge, the first Theorem about extension of isometries (see [6]):

Mankiewicz's Theorem (1972): Let X and Y be normed spaces, $F_X \subset X$ and $F_Y \subset Y$ be closed bodies. If $\tau : F_X \to F_Y$ is an onto isometry, then it is the restriction of an affine isometry $\widetilde{\tau} : X \to Y$.

Mazur-Ulam Property

Our problem is to determine whether every onto isometry between spheres extends or not. A closely related problem is whether, fixing a space $(X, \|\cdot\|_X)$, every onto isometry $\tau : S_X \to S_Y$ extends or not. This has led to:

Definition: A normed space $(X, \|\cdot\|_X)$ has the Mazur-Ulam Property if every onto isometry $\tau : S_X \to S_Y$ extends to a (linear, onto) isometry $\tilde{\tau} : X \to Y$.

Extension of isometries

A weaker related problem is to find conditions that ensure that, for some kind of spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$, every onto isometry $\tau : S_X \to S_Y$ is the restriction of some linear isometry $\widetilde{\tau} : X \to Y$. As for example:

Theorem ([4]): If we consider $\mathbb{R}^2$ endowed with two p-norms, say $\|\cdot\|_X = \|\cdot\|_p$ and $\|\cdot\|_Y = \|\cdot\|_q$ with $p, q \in (1, \infty)$, and there is an isometry $\tau : S_X \to S_Y$, then $q = p$ and τ is either the identity, a symmetry or a rotation. Anyway, τ extends to a linear isometry defined on X.

Tingley's problem again

What we are dealing with is, thank to Mazur, Ulam and Mankiewicz, equivalent to each of the following:

Question: Is every onto isometry $\tau : S_X \to S_Y$ the restriction of an isometry $\widetilde{\tau} : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$?

Question: Is every onto isometry $\tau : S_X \to S_Y$ the restriction of an isometry $\tau : B_X \to B_Y$?

The natural extension

The first idea that arose when dealing with Tingley's Problem was quite simple:
If $\tau : S_X \to S_Y$ has a linear extension $\widetilde{\tau} : X \to Y$, of course this extension must fulfil $\widetilde{\tau}(\lambda x) = \lambda \tau(x)$ for every $\lambda \geq 0$. So, the idea is to take this *natural extension* $\widetilde{\tau}$ and prove that it is linear.

Some results obtained with this approach

Theorem: Let $\tau : S_X \to S_Y$ be a surjective isometry between the unit spheres of two compact C^*-algebras. Then there exists a (unique) surjective real linear isometry $\widetilde{\tau} : X \to Y$ such that $\widetilde{\tau}(x) = \tau(x)$ for every $x \in S_X$.

Theorem: Let $\tau : S_X \to S_Y$ be a surjective isometry between the unit spheres of two von Neumann algebras. Then there exists a surjective real linear isometry $\widetilde{\tau} : X \to Y$ that restricted to S_X is τ.

Theorem: Let $(H_i)_{i\in I}$ and $(K_j)_{j\in J}$ be two families of complex Hilbert spaces. Suppose $\Delta : S\left(\bigoplus_j^{l_\infty} B(K_j)\right) \to S\left(\bigoplus_i^{l_\infty} B(H_i)\right)$ is a surjective isometry. Then there exists a real linear isometry

$$T : \bigoplus_j^{l_\infty} B(K_j) \to \bigoplus_i^{l_\infty} B(H_i)$$

satisfying $T_{|S(E)} = \Delta$.

Restriction of linear means linear

Let $\tau : S_X \to S_Y$ be any map. If is it the restriction of some linear map $\widetilde{\tau} : X \to Y$, in particular $\tau(\lambda x + \lambda' x')$ must be $\lambda\tau(x) + \lambda'\tau(x')$ whenever $x, x', \lambda x + \lambda' x' \in S_X$.

Linear isometric isomorphisms *from* $\mathbb{R}^n$

We may take coordinates. Namely, given an n-dimensional space $(X, \|\cdot\|_X)$ and a basis $\mathcal{B}_X = \{x_1, \ldots, x_n\} \subset X$, we may identify $(X, \mathcal{B}_X)$ with $\mathbb{R}^n$ endowed with its usual basis $\mathcal{B}_n = \{e_1, \ldots, e_n\}$ by defining a norm in $\mathbb{R}^n$ as

$$\|(\lambda_1, \ldots, \lambda_n)\|'_X = \|\lambda_1 x_1 + \ldots + \lambda_n x_n\|_X$$

and $\phi_X : \mathbb{R}^n \to X$ given by

$$\phi_X(\lambda_1, \ldots, \lambda_n) = \lambda_1 x_1 + \ldots + \lambda_n x_n.$$

Linear isometric automorphisms *of* $\mathbb{R}^n$

If $\mathcal{B}_X = \{x_1, \ldots, x_n\} \subset S_X$ is a basis, $\tau : S_X \to S_Y$ is an isometry and $\mathcal{B}_Y = \{\tau(x_1), \ldots, \tau(x_n)\} \subset S_Y$ is also a basis, then we may identify both $(X, \mathcal{B}_X)$ and $(Y, \mathcal{B}_Y)$ with $\mathbb{R}^n$ by means of linear isometries and we have an onto isometry τ' between the spheres (in $\mathbb{R}^n$) of $\|\cdot\|'_X$ and $\|\cdot\|'_Y$ that leaves fixed every $e_i \in \mathcal{B}_n$, i.e.,

$$\tau'(1,0,\ldots,0) = (1,0,\ldots,0), \quad \tau'(0,1,\ldots,0) = (0,1,\ldots,0)\ldots$$

This means that τ and τ' are linear if and only if τ' is the identity.

Is τ' the identity?

The way that we approach Tingley's Problem is:
Question: Let $(\mathbb{R}^n, \|\cdot\|_X)$ and $(\mathbb{R}^n, \|\cdot\|_Y)$ be normed spaces such that there is an isometry $\tau : S_X \subset \mathbb{R}^n \to S_Y \subset \mathbb{R}^n$ between their spheres such that $\tau(e_i) = e_i$ for every $e_i \in \mathcal{B}_n$. Is τ the identity? In particular, are S_X and S_Y the same subset of $\mathbb{R}^n$?

Spheres and their geometry

For every onto isometry defined on the sphere S_X of a normed space to be linear it is sufficient that S_X has enough metric invariants to distinguish it from every other sphere.

Hypotheses and notations

We will suppose that

- X and Y are two-dimensional spaces,
- $\tau : S_X \to S_Y$ is an isometry,
- $\mathbf{r}_X : \mathbb{R} \to S_X$ and $\mathbf{r}_Y : \mathbb{R} \to S_Y$ are arc-length parameterizations of S_X and S_Y such that $\mathbf{r}_Y(t) = \tau(\mathbf{r}_X(t))$ for every $t \in \mathbb{R}$.
- L is the half-length of S_X (of S_Y, too).

By Tingley's result, we have $\tau(-x) = -\tau(x)$ for every $x \in S_X$.

Our goal

We will show that when we take coordinates with respect to appropriate bases $\mathcal{B}_X = \{x, x'\} \subset S_X$ and $\mathcal{B}_Y = \{y, y'\} \subset S_Y$, τ is the identity.

Equivalently, that $\tau(\lambda x + \mu x') = \lambda y + \mu y'$ for every λ, μ such that $\lambda x + \mu x' \in S_X$.

Preliminar observations

Once we have taken coordinates, we may consider the subset $S_X \cap S_Y$, that is always closed in $\mathbb{R}^2$ and also in S_X. As S_X is conected, if $S_X \cap S_Y$ is open in S_X then $S_X \cap S_Y = S_X$, i.e., $S_Y = S_X$.

Thanks to any of the versions of the Monotonicity Lemma, we get that every $\overline{x} \in S_X$ is determined by the distances

$$\|\overline{x} - x\|_X, \ \|\overline{x} - x'\|_X, \ \|\overline{x} + x\|_X, \ \|\overline{x} + x'\|_X.$$

This implies that the only *autoisometry* $\tau : S_X \to S_X$ such that $\tau(x) = x, \tau(x') = x'$ is the identity. Joining all these facts, we obtain that if $S_X \cap S_Y$ is open in S_X then τ is the identity.

Segments

This case has been solved in [4].
Let, for each $x \in S_X$, $\mathcal{D}(x) = \{x' \in S_X : \|x - x'\|_X = 2\}$.
This subset consists of the segments in S_X that contain $-x$.
As $\mathcal{D}(x)$ is defined by means of distances between points of S_X, one has $\mathcal{D}(\tau(x)) = \tau(\mathcal{D}(x))$.
So, every segment containing x goes to another segment containing $\tau(x)$. In particular, the amount of segments in S_X is the same as in S_Y.

How to solve this case

Take some segments $[x_1, x_2] \subset S_X$ and $[y_1, y_2] \subset S_Y$, where $y_1 = \tau(x_1), y_2 = \tau(x_2)$.

Right now, I need to draw a little.

The natural parameterization

This case has been solved in [2].
I am very lucky because Tarás already gave a seminar to explain this case.
Suppose that the norms are strictly convex and smooth and, moreover, the natural parameterizations' derivatives are not only continuous but *absolutely continuous*.

This is exactly what we need to ensure that $\mathbf{r}_X'(t)$ exists at every $t \in \mathbb{R}$, and that $\mathbf{r}_X'(t) = \int_0^t \mathbf{r}_X''(s)\,\mathrm{d}s + \mathbf{r}_X'(0)$. (Analogously with $\mathbf{r}_Y'$).

Given S_X and S_Y, and the natural parameterizations $\mathbf{r}_X, \mathbf{r}_Y$, we may take the bases $\mathcal{B}_X = \{\mathbf{r}_X(0), \mathbf{r}_X'(0)\}$ and $\mathcal{B}_Y = \{\mathbf{r}_Y(0), \mathbf{r}_Y'(0)\}$.

How to use a differential equation

The proof of the fact that, under these circumstances, $\mathbf{r}_X(t) = \mathbf{r}_Y(t)$ for every $t \in \mathbb{R}$ comes from the following:

• As $\mathbf{r}_X(t)$ and $\mathbf{r}'_X(t)$ are linearly independent, for each t there exist unique $\rho_X(t), \sigma_X(t); \rho_Y(t), \sigma_Y(t) \in \mathbb{R}$ such that

$$\mathbf{r}''_X(t) = -\rho_X(t)\mathbf{r}'_X(t)+\sigma_X(t)\mathbf{r}_X(t); \quad \mathbf{r}''_Y(t) = -\rho_Y(t)\mathbf{r}'_Y(t)+\sigma_Y(t)\mathbf{r}_Y(t).$$

• Given well-behaved functions $\rho_X, \sigma_X, \rho_Y, \sigma_Y$, there is exactly one solution ($\mathbf{r}_X(t)$ and $\mathbf{r}_Y(t)$) to each differential equation. This is so because we have the right amount of initial conditions: $\mathbf{r}_X(0) = (1,0) = \mathbf{r}_Y(0), \mathbf{r}'_X(0) = (0,1), \mathbf{r}'_Y(0)$.

• The functions ρ_X, σ_X can be computed by means of distances between points of S_X!! So, we obtain $\rho_Y = \rho_X, \sigma_Y = \sigma_X$ and the uniqueness of the solution gives $\mathbf{r}_X = \mathbf{r}_Y$.

Recognizing non-smoothness from close

This case has been solved in [5].

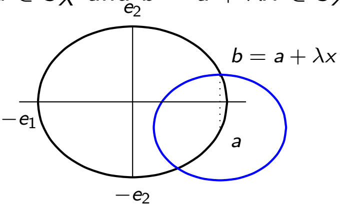
Given $x \in S_X$, the following are mutually equivalent:

- The norm $\|\cdot\|_X$ is differentiable at x.
- The sphere S_X is differentiable at x.
- If $\mathbf{r}_X(t_0) = x$ (and $\mathbf{r}_X(t_0 + L) = -x$), then the map $t \mapsto \|\mathbf{r}_X(t_0 - t) - \mathbf{r}_X(t_0 + L + t)\|_X$ is differentiable at $t = 0$.

The third condition is preserved by the isometry τ, so S_X is differentiable at x if and only if S_Y is differentiable at $\tau(x)$.

Partial derivatives and linear combinations

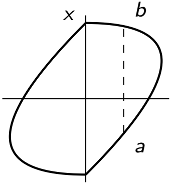
Let $x = \mathbf{r}_X(t_0) \in S_X$ and take the basis $\mathcal{B}_X = \{-\mathbf{r}'_X(t_0), x\}$. Take, furthermore, some $a \in S_X$ and $b = a + \lambda x \in S_X$.



With $F(\alpha, \beta) = \|b + \alpha e_1 + \beta e_2 - a\|_X$, we have $\frac{\partial F}{\partial e_1}(0,0) = 0$; $\frac{\partial F}{\partial e_2}(0,0) = 1$. So, if the derivative of $\mathbf{r}_X$ at b is (b'_1, b'_2) then $\frac{d}{\mathrm{d}t}\|b + t(b'_1, b'_2) - a\|_X(0) = b'_2$. So, b'_2 is the speed of growing of the distance to a when $\mathbf{r}_X$ passes through b.

Recognizing non-smoothness from far

If S_X is not differentiable at x, then we have something like this:



As before, we have that b_2' is the rate of growth of $\|\mathbf{r}_X(t) - a\|_X$ when $\mathbf{r}_X(t)$ approaches b from right and down. But this is not how $\|\mathbf{r}_X(t) - a\|_X$ grows when $\mathbf{r}_X(t)$ approaches b from left and above. This means that we can recognize when $(a-b)/\|a-b\|_X$ is a point of differentiability of S_X, at least when S_X is smooth in b.

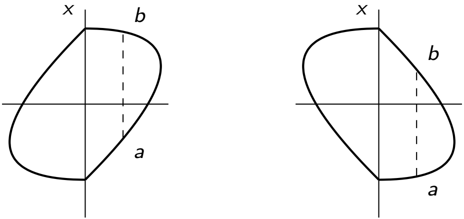
Special directions

A point $x \in S_X$ is called *special* if for any bijective isometry $\tau : S_X \to S_Y$ to the unit sphere of a Banach space Y and any points $y, z \in S_X$ with $y - z = \|y - z\| \cdot x$ we have

$$\tau(y) - \tau(z) = \|\tau(y) - \tau(z)\| \cdot \tau(x) = \|y - z\| \cdot \tau(x).$$

When there is no basis

In this case, when there is exactly one pair of nondifferentiability points $x, -x$ in S_X, we have that x is special. Moreover, we may compute the change of coordinates needed to transform the basis that makes S_X *arrive at* $x = (0, 1)$ horizontally into the basis that makes S_X *leave* $x = (0, 1)$ horizontally.



The linear isometry (a.k.a. change of bases)

If T is the change of bases from $\mathcal{B}_X = \{\mathbf{r}_{-,X}\}$ to $\overline{\mathcal{B}}_X = \{\mathbf{r}_{+,X}\}$ and as A is its matrix with respect to $\mathcal{B}_X$ then (as $T(0,1) = (0,1)$) we have $A = \begin{pmatrix} \alpha & 0 \\ \beta & 1 \end{pmatrix}$. Moreover, we can determine α and β by means of distances, so the change of bases is the same for S_X and S_Y.

The second coordinates are known

To finish this sub-case, denoting (b_1', b_2') the derivative of $\mathbf{r}_X$ at b and $T(b_1', b_2') = (\overline{b}_1', \overline{b}_2')$, we can determine b_2' and $\overline{b}_2'$. Namely, b_2' is the speed of growth of $\|\mathbf{r}_X(t) - a\|$ when $\mathbf{r}_X(t)$ approaches b by the right down side and $\overline{b}_2'$ is the same but approaching b from left and above.

Some basic linear algebra

We have $(\overline{b}_1', \overline{b}_2') = (\alpha b_1', \beta b_1' + b_2')$ and b_2', $\overline{b}_2'$ and β are known, so we can determine $b_1', \overline{b}_1'$.

This means that we know both coordinates of the derivative of every point of S_X and these coordinates are the same in S_Y. With this, the result follows (but, surprisingly, only when there is exactly one pair of nondifferentiability points in S_X).

The importance of finiteness

We know that $(\tau(a)-\tau(b))/\|\tau(a)-\tau(b)\|_Y$ is a nondifferentiability point of S_Y if and only if S_X is not smooth in $(a-b)/\|a-b\|_X$.
We are in the Case 3: Piecewise smooth spaces. This means that there are only finitely many points of nondifferentiability in S_X (in S_Y, too).
Under these conditions, every nondifferentiability pont in S_X is special.

When there is a basis

If $e_1, e_2 \in S_X$ are linearly independent and both of them are *special*, then we can take coordinates with respect to $\mathcal{B}_X = \{e_1, e_2\}$ and $\mathcal{B}_Y = \{\tau(e_1), \tau(e_2)\}$ and get

$$x - x' = (\lambda, 0) \iff \tau(x) - \tau(x') = (\lambda, 0);$$

$$x - x' = (0, \mu) \iff \tau(x) - \tau(x') = (0, \mu).$$

With due care, this is enough to prove the result in this case: every isometry $\tau : S_X \to S_Y$ is linear.

Some surprise

In this setting, the result is even more general:

Theorem: If S_X and S_Y are piecewise differentiable, $C_X \subset X$ and $C_Y \subset Y$ are piecewise differentiable convex Jordan curves and $\tau : C_X \to C_Y$ is an isometry, then τ is affine and X and Y are linearly isometric, provided there is a basis $x, x' \in S_X$ of nondifferentiability points.

A brilliant idea

This case has been solved in [3].
There was a clear path towards the result in non-smooth spaces, and then I had the most clever idea I have ever had as a mathematician... Ask Tarás Banakh for help.

The lack of importance of finiteness

It turns out that Tarás found a way to measure the angles in the spheres (jumps of $\mathbf{r}'_X$) and this was enough to show that every nondifferentiability point of every two-dimensional sphere is special.

S_X is C^1 but $\mathbf{r}'_X$ is not absolutely continuous

This case has been solved in [1].
The idea is that the jumps of $\mathbf{r}''_X$ can also be measured by means of the distances. So, every point where S_X is not twice differentiable is special and the result holds... because there are infinitely many points where $\mathbf{r}''_X$ does not exist.
It is not exactly this way, but it is close.

I hope I'll be as lucky with this case as with the second one and Tarás will give a seminar to explain it, too.

Dzięki

Thank you very much for your attention.

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