Nonlinear geometry of operator spaces

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Collaborators

This talk contains joint work with

- Javier Alejandro Chávez-Domínguez (University of Oklahoma)
- Thomas Sinclair (Purdue University)

Looking at Banach spaces as metric spaces

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A map $f : X \to Y$ between metric spaces (X, d) and (Y, ∂) is an **isometry** if

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$$\partial(f(x), f(y)) = d(x, y) \text{ for all } x, y \in X.$$

Theorem (Mazur and Ulam, 1932)

If f is a surjective isometry between Banach spaces and $f(0) = 0$, then f is $\mathbb{R}$-linear.

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Let (X, d) and (Y, ∂) be metric spaces, $f : X \to Y$ be a map and consider the modulus

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- Let G be a finitely generated group. If S and F are finite generating sets, then the Cayley graphs $\mathrm{Cay}(G, S)$ and $\mathrm{Cay}(G, F)$ are coarsely equivalent.

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- If X is infinite dimensional, any two nets of X are Lipschitzly equivalent.

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Theorem (Johnson, Lindenstrauss, Schechtman, 1996)

If a Banach space is coarsely equivalent to ℓ_p, for $p \in (1, \infty)$, then it is isomorphic to it.

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$$[x_{ij}] = \begin{bmatrix} x_{11} & \dots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \dots & x_{nn} \end{bmatrix}.$$

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- In particular, if X is a closed linear subspace of $\mathcal{B}(H)$, then $\mathrm{M}_n(X) \subset \mathrm{M}_n(\mathcal{B}(H))$, so $\mathrm{M}_n(X)$ inherits a norm from $\mathrm{M}_n(\mathcal{B}(H))$.

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An **operator space** is a closed linear subspace of $\mathcal{B}(H)$ for some Hilbert space H. So each $\mathrm{M}_n(X)$ is a Banach space.

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What is the difference between Banach spaces and operator spaces then?

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- Given a linear map $f : X \to Y$ between operator spaces and $n \in \mathbb{N}$, $\|f_n\|_n$ is the norm of $f_n : \mathrm{M}_n(X) \to \mathrm{M}_n(Y)$ and

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- If $\|f\|_{\mathrm{cb}} < \infty$, f is **completely bounded** and, if $\|f_n\|_n \leqslant 1$ for all $n \in \mathbb{N}$, f is a **complete contraction.**
- The set of all completely bounded maps $X \to Y$ is denoted by $\mathrm{CB}(X, Y)$.
- f is a **complete isomorphism** if both f and f^{-1} are completely bounded and a **complete isometry** if both f and f^{-1} are complete contractions.

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  So $T([a_{i,j}]) = [a_{j,i}]$ for all $a = [a_{i,j}] \in \mathcal{B}(\ell_2)$ where $a_{i,j} = \langle a\delta_j, \delta_i \rangle$.

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  For $i, j \in \mathbb{N}$, let e_{ij} be the rank 1 partial isometry sending δ_j to δ_i.

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  Then, for each $n \in \mathbb{N}$, we have

$$\left\| \begin{bmatrix} e_{11} & 0 & \dots & 0 \\ e_{12} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ e_{1n} & 0 & \dots & 0 \end{bmatrix} \right\|_n = 1 \text{ and } \left\| \begin{bmatrix} e_{11} & 0 & \dots & 0 \\ e_{21} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ e_{n1} & 0 & \dots & 0 \end{bmatrix} \right\|_n = \sqrt{n}$$

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- The norm map $N : x \in \mathcal{B}(\ell_2) \to \|x\| \in \mathbb{R}$ is not completely bounded.
  Indeed, if $a \in \mathrm{M}_n(\mathcal{B}(\ell_2))$ is the matrix on the left above, then
  $\|a\|_n = 1$ but $\mathrm{Lip}(N_n) = \sqrt{n}$ for all $n \in \mathbb{N}$.

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$$\|[f(x_{ij})]\|_n = \sup\{\langle [f(x_{ij})]\alpha, \beta\rangle \mid \alpha, \beta \in B_{\ell_2^n}\}.$$

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  $$\|[f(x_{ij})]\|_n = \sup\{\langle [f(x_{ij})]\alpha, \beta\rangle \mid \alpha, \beta \in B_{\ell_2^n}\}.$$
  If $\alpha = (\alpha_i)_{i=1}^n$ and $\beta = (\beta_i)_{i=1}^n$, then
  $$\langle [f(x_{ij})]\alpha, \beta\rangle = \sum_{i,j} \alpha_j \bar{\beta}_i f(x_{ij}) = f\Big(\sum_{i,j} \alpha_j \bar{\beta}_i x_{ij}\Big) \leqslant \|f\| \Big\| \sum_{i,j} \alpha_j \bar{\beta}_i x_{ij} \Big\|.$$

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  Let $n \in \mathbb{N}$ and $[x_{ij}] \in \mathrm{M}_n(X)$. Then $[f(x_{ij})] \in \mathrm{M}_n$, so

  $$\|[f(x_{ij})]\|_n = \sup\{\langle [f(x_{ij})]\alpha, \beta\rangle \mid \alpha, \beta \in B_{\ell_2^n}\}.$$

  If $\alpha = (\alpha_i)_{i=1}^n$ and $\beta = (\beta_i)_{i=1}^n$, then

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  Notice that

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So, $\|f_n\|_n \leqslant \|f\|$ for all $n \in \mathbb{N}$.

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Theorem (Z.-J. Ruan, 1988 and 2003)

Let X be a vector space and $(\|\cdot\|_n)_n$ be a sequence of norms on $(\mathrm{M}_n(X))_n$. Then X is completely isometrically isomorphic to a subspace of $\mathcal{B}(H)$, for some Hilbert space H, if and only if $(\|\cdot\|_n)_n$ satisfies (R1) and (R2).

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- Let X be a Banach space and consider its canonical embedding into the abelian C^*-algebra $C(B_{X*})$. The canonical isomorphim between $M_n(C(B_{X*}))$ and $C(B_{X*}, \mathrm{M}_n)$ gives us an operator space structure on X.

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- If A is a C^*-algebra, so is each $\mathrm{M}_n(A)$. So, C^*-algebras have unique operator space structures.

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While all Banach spaces are locally reflexive, this is not the case for operator spaces. E.g.: $\mathcal{K}(\ell_2)$. On the other hand, any von Neumann algebra predual is locally reflexive (**Effros, Junge, and Ruan, 2000**).

Nonlinear geometry of operator spaces

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Theorem (Braga and Chávez-Domínguez, 2021)

If f is a completely coarse map between operator spaces and $f(0) = 0$, then f is $\mathbb{R}$-linear.

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Theorem (Braga and Chávez-Domínguez, 2021)

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Given an operator space X, there is an operator space structure on X, $\mathrm{MIN}_n(X)$, which agrees with the one of X up to the n-level and so that

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- The nonlinear embedding must be strong enough to recover linear aspects of the operator structures

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Definition

Let X and Y be operator spaces. We say that X **almost completely coarsely embeds into** Y if there is a sequence $(f^n : X \to Y)_n$ so that the amplifications $(f^n_n : \mathrm{M}_n(X) \to \mathrm{M}_n(Y))_n$ are equi-coarse embeddings, i.e., there are functions $\omega, \rho : [0, \infty) \to [0, \infty)$ so that $\lim_{t\to\infty} \rho(t) = \infty$ and so that

$$\rho(\|[x_{ij}] - [y_{ij}]\|_n) \leqslant \|f^n_n([x_{ij}]) - f^n_n([y_{ij}])\|_n \leqslant \omega(\|[x_{ij}] - [y_{ij}]\|_n)$$

for all $n \in \mathbb{N}$ and all $[x_{ij}], [y_{ij}] \in \mathrm{M}_n(X)$.

Is this an “interesting” notion? We need:

I The nonlinear embedding must be strong enough to recover linear aspects of the operator structures

II The nonlinear embedding must be strictly weaker than almost complete isomorphic embeddability

At least Item I holds!

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Given a Banach space Y and an ultrafilter $\mathcal{U}$ on $\mathbb{N}$, the **ultraproduct** of Y is defined by

$$Y^{\mathbb{N}}/\mathcal{U} = \Big\{(y_k)_k \in Y^{\mathbb{N}} \mid \sup_k \|y_k\| < \infty\Big\}/\Big\{(y_k)_k \in Y^{\mathbb{N}} \mid \lim_{k,\mathcal{U}} \|y_k\| = 0\Big\}.$$

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The natural operator structure of $Y^{\mathbb{N}}/\mathcal{U}$: given $[(x_{ij}^{(k)})_k] \in \mathrm{M}_n(Y^{\mathbb{N}}/\mathcal{U})$,

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Proposition (Braga and Chávez-Domínguez, 2021)

If an operator space X almost completely coarsely embeds into an operator space Y, then X completely $\mathbb{R}$-isomorphically embeds into $Y^{\mathbb{N}}/\mathcal{U}$ for any nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$.

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Sketch of the proof.

Given $(f^n : X \to Y)_n$, $F(x) = (f^n(x))_n$ is completely coarse. □

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Theorem (Braga and Chávez-Domínguez, 2021)

If an infinite dimensional $\mathbb{C}$*-operator space* X *almost completely coarsely embeds into* OH*, then* X *is completely* $\mathbb{C}$*-isomorphic to* OH*.*

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- If $K = 1$, $(f^n : X \to Y)_n$ is an **almost complete isometric embedding**.

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We mimic $\mathcal{F}(X)'s$ construction for our setting.

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Definition

We denote the operator space $(\mathrm{Lip}_0(X, Y), (\|\cdot\|_{\mathrm{Lip},n,k})_{k\in\mathbb{N}})$ defined above by $\mathrm{Lip}_0^n(X, Y)$. **Notice:** By Ruan's characterization, we are OK.

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Proposition

If $X \subset \mathcal{B}(H)$, then the canonical map

$$\iota : \mathcal{B}(H)^* \to \mathrm{Lip}_0^n(X, \mathbb{C})$$

given by $\iota(a) = a\restriction X - a(x_0)$ is a complete contraction.

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Definition

We define the ***n*-Lipschitz-free operator space of** (X, x_0) as the Banach space

$$\mathcal{F}^n(X) = \overline{\mathrm{span}}\Big\{\delta_x \in \mathrm{Lip}_0^n(X, \mathbb{C})^* \mid x \in X\Big\}$$

together with the operator space structure inherited from $\mathrm{Lip}_0^n(X, \mathbb{C})^*$.

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There are operator spaces X and Y so that X almost completely isometrically embeds into Y but so that X does not $\mathbb{R}$-isomorphically embed into Y.

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There are operator spaces X and Y so that X almost completely isometrically embeds into Y but so that X does not $\mathbb{R}$-isomorphically embed into Y.

The separable case remains open.

Left inverse of δ_X^n

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$$\Big|x^*\Big(\sum_i a_i x_i\Big)\Big| = \Big|\Big(\sum_i a_i \delta_{x_i}\Big)(x^*)\Big| \leqslant \Big\|\sum_i a_i \delta_{x_i}\Big\| \|x^*\|.$$

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Corollary (Braga, Chávez-Domínguez, and Sinclair, 2021)

Let (X, x_0) be a pointed operator metric space and Y be an operator space. For any $L \in \mathrm{Lip}_0(X, Y)$ there is a unique linear map $\bar{L} : \mathcal{F}^n(X) \to Y$ such that $L = \bar{L}\delta_X^n$ and $\|\bar{L}\|_{\mathrm{cb}} = \mathrm{Lip}(L_n)$.

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The Lipschitz-free space of a tree on $k+1$ vertices can be isometrically identified with ℓ_1^k as Banach spaces.

Proposition (Braga, Chávez-Domínguez, and Sinclair, 2021)

Let $(T,0)$ be a rooted tree with $k+1$ vertices, say $T=\{0,1,\ldots,k\}$. Consider the isometry $T\to\ell_1^k$ given by

$$0\in T\mapsto 0\in\ell_1^k \ \textit{ and } \ j\in T\backslash\{0\}\mapsto\sum_{0\prec i\preccurlyeq j} e_i\in\ell_1^k.$$

Consider T as an operator metric space with the structure induced by this isometry and the maximal operator space structure on ℓ_1^k. Then for any $n\in\mathbb{N}$, $\mathcal{F}^n(T)$ is completely isometric to $\mathrm{MAX}(\ell_1^k)$.

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Sketch of the proof.

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So, $\tilde{L}$ is a bijection, $\|\tilde{L}\|_{\mathrm{cb}} = 1$, and

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$$\|\tilde{L}^{-1} : \mathrm{MAX}(\ell_1^k) \to \mathcal{F}^n(T)\|_{\mathrm{cb}} = \|\tilde{L}^{-1} : \ell_1^k \to \mathcal{F}^n(T)\|$$

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From the classic setting, $\|\tilde{L}^{-1} : \ell_1^k \to \mathcal{F}^1(T)\| = 1$.

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As $\mathrm{Lip}(f) \leqslant \mathrm{Lip}(f_n)$, the identity $\mathrm{Lip}_0^n(T, \mathbb{C}) \to \mathrm{Lip}_0(T, \mathbb{C})$ has norm at most 1. So, the same holds for $\mathrm{Id} : \mathcal{F}^1(T) \to \mathcal{F}^n(T)$.

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Let X be an operator space and $n \in \mathbb{N}$. We say that X has the ***n*-isometric Lipschitz-lifting property** if there exists a linear n-contraction $T : X \to \mathcal{F}^n(X)$ such that $\beta^n_X T = \mathrm{Id}_X$.

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Every separable operator space has the n-isometric Lipschitz-lifting property for all $n \in \mathbb{N}$.

Theorem (Braga, Chávez-Domínguez, and Sinclair, 2021)

If a separable operator space X almost completely isometrically embeds into an operator space Y, then X almost completely $\mathbb{R}$-linearly isometrically embeds into Y.

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$$\|\mu\|_{\mathcal{F}(X)} = \inf\Big\{ \sum_{i=1}^{k} |a_i| d(x_i, y_i) \mid \mu = \sum_{i=1}^{k} a_i(\delta_{x_i} - \delta_{y_i})\Big\}.$$

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Theorem (Braga, Chávez-Domínguez, and Sinclair, 2021)

For any $m \in \mathbb{N}$ and $\mu \in \mathrm{M}_m(\mathrm{span}\{\delta_x\}_{x\in X})$ we have

$$\|\mu\|_{\mathrm{M}_m(\mathcal{F}^n(X))} = \inf\Big\{ \|\alpha\|\|\beta\| \max_{1\leqslant \ell \leqslant N} |c_\ell| \|[x^\ell_{ij} - y^\ell_{ij}]\| \Big\}$$

where the infimum is taken over all $N \in \mathbb{N}$ and all representations $\mu = \alpha \cdot D \cdot \beta$ where $\alpha \in \mathrm{M}_{m,Nn}$ and $\beta \in \mathrm{M}_{Nn,m}$, and $D \in \mathrm{M}_N(\mathrm{M}_n(F))$ is a diagonal matrix whose diagonal entries are of the form $c_\ell[\delta_{x^\ell_{ij}} - \delta_{y^\ell_{ij}}]_{ij}$.

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Recall: A map $f : X \to Y^*$ is **Gateaux w^*-$\mathbb{R}$-differentiable at $x \in X$** if for all $a \in X$ the limit

$$D^* f_x(a) = w^*\text{-}\lim_{\lambda \to 0} \frac{f(x + \lambda a) - f(x)}{\lambda}$$

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If X and Y are separable, a Lipschitz function $u : X \to Y^*$ is Gateaux w^*-$\mathbb{R}$-differentiable “almost everywhere” **(Heinrich and Mankiewicz, 1982).**

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  Indeed, if $[a_{ij}] \in \mathrm{M}_n(X)$ then $D^* f_x([a_{ij}])$ is in $\mathrm{M}_n(Y^*) = \mathrm{CB}(Y, \mathrm{M}_n)$. So, $\|D^* f_x([a_{ij}])\|_{\mathrm{cb}}$ equals

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$$\begin{aligned}\|D^* f_x([a_{ij}])\|_{\mathrm{cb}} &\leqslant \lim_{\lambda\to 0}\frac{1}{\lambda}\cdot\Big\|f_n([x+\lambda a_{ij}]) - f_n([x])\Big\|_n \\ &\leqslant \mathrm{Lip}(f_n)\cdot\|[a_{ij}]\|_n.\end{aligned}$$

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  Hence, $\|(D^* f_x)_n\|_n \leqslant \mathrm{Lip}(f_n)$.

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Let X and Y be operator spaces and X be separable. If X almost completely Lipschitzly embeds into Y^*, then X almost completely $\mathbb{R}$-isomorphically embeds into Y^*.

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Corollary (Braga, Chávez-Domínguez, and Sinclair, 2021)

Let X and Y be operator spaces and assume that X almost completely Lipschitzly embeds into Y^*. Then X almost completely $\mathbb{C}$-linearly embeds into $Y^* \oplus \overline{Y}^*$.

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(and now I am going back to bed...)