Approximate convexity in the Takagi class

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Abstract

For a sequence $a = (a_i)_{i\in\mathbb{N}} \subset \mathbb{R}_+$ we consider Takagi-like function of the form

$$T[a](x) := \sum_{i=1}^{\infty} a_i \mathrm{dist}(x; \tfrac{1}{2^{i-1}}\mathbb{Z}) \quad \text{for } x \in \mathbb{R}.$$

Assume that there exists a $q > 1/2$ such that

$$a_{i+1} \geq q a_i \quad \text{for } i \in \mathbb{N}.$$

Our main results reads as follows:

- $T[a]$ is paraconvex if and only if $\lim\limits_{i\to\infty} a_i = 0$;
- $T[a]$ is strongly paraconvex if and only if $\sum\limits_{i=1}^{\infty} a_i < \infty$.

Using the above statements we construct paraconvex functions which are almost everywhere non-differentiable.

1 Introduction

In the year 1903 T. Takagi [16] introduced the function

$$T(x) := \sum_{n=1}^{\infty} \mathrm{dist}(x; \tfrac{1}{2^{n-1}}\mathbb{Z}) \quad \text{for } x \in \mathbb{R},$$

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which gives a simple example of a continuous nowhere differentiable function. Since then the Takagi function and its generalizations of the form

$$T[a](x) := \sum_{n=1}^{\infty} a_n \mathrm{dist}(x; \tfrac{1}{2^{n-1}}\mathbb{Z}) \quad \text{for } x \in \mathbb{R}, \tag{1}$$

where $a = (a_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ has been applied in various parts of mathematics, in particular in the theory of fractals [3], approximate convexity and functional equations [1, 5, 6, 7, 14, 15] or special functions theory [4, 7, 8]. For the survey on the results of the functions of Takagi type we refer the reader to [7]. It is worth mentioning that by [4, Theorem 2.2] $T[a]$ is a well-defined real-valued function if and only if $\sum\limits_{n\in\mathbb{N}} 2^n|a_n| < \infty$.

Our aim is to show that the functions of Takagi class can serve as an important source of examples and counterexamples for paraconvex and semiconvex functions. We will show that the Takagi functions have a large variety of properties related to approximate convexity. To explain our main results, we need to recall some notions of approximate convexity [10, 11, 12, 13, 17].

Definition 1.1. Let $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing function such that $\lim\limits_{r\to 0^+} \gamma(r)/r = 0$.

Let V be a convex subset of a normed space. We say that a function $f : V \to \mathbb{R}$ is $\gamma(\cdot)$*-paraconvex* if

$$\mathcal{C}f(x, y; t) := f(tx+(1-t)y)-tf(x)-(1-t)f(y) \le \gamma(\|x-y\|) \quad \text{for } x, y \in V, t \in [0, 1]. \tag{2}$$

We call f *strongly* $\gamma(\cdot)$*-paraconvex* if

$$\mathcal{C}f(x, y; t) \le \min(t, 1-t)\gamma(\|x-y\|) \quad \text{for } x, y \in V, t \in [0, 1]. \tag{3}$$

We will say that f is *(strongly) paraconvex* if there exists a respective function γ such that f is (strongly) $\gamma(\cdot)$-paraconvex.

An almost equivalent notion to strong paraconvexity is the notion of semiconvexity [2]. In fact on open convex sets semiconvexity is equivalent to strong paraconvexity [18]. Let us mention that paraconvex, strongly paraconvex and semiconvex functions play an important role in the study of real-valued functions on normed spaces [2, 5, 6, 9, 10, 11, 12, 13, 17, 18]. Important directions in the study of paraconvexity and semiconvexity are the following:

- prove that (under some additional assumptions) paraconvex functions are strongly paraconvex [11, 13].
- show that strongly paraconvex functions are almost everywhere differentiable, see for example [12, 13, 17].

In this paper we deal with, to some extent, dual problems:

- does there exist a paraconvex function $f : [0,1] \to \mathbb{R}$ which is not strongly paraconvex?
- is every paraconvex function $f : [0,1] \to \mathbb{R}$ almost everywhere differentiable?

We answer the above questions negatively by giving a necessary and sufficient conditions for $T[a]$ to be paraconvex or strongly paraconvex. This jointly with the Theorem of Kôno [8, Theorem 2] implies that

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \mathrm{dist}(x; \tfrac{1}{2^{n-1}}\mathbb{Z})$$

is an example of a paraconvex function which is differentiable only on a set of measure zero (and consequently is not strongly paraconvex).

2 Preliminary results

In this section we prove a list of technical lemmas. We begin with an obvious but important result.

Proposition 2.1. *Let V be a convex subset of a normed space and let $f : V \to \mathbb{R}$ be a Lipschitz function. Then*

$$|\mathcal{C}f(x,y;t)| \le 2t(1-t)\mathrm{lip}(f)\|x-y\| \quad \textit{for } x,y \in V, t \in [0,1].$$

Proof. For $x,y \in V, t \in [0,1]$ we have

$$|\mathcal{C}f(x,y;t)| = t|f(x) - (tx + (1-t)y)| + (1-t)|f(tx+(1-t)y) - f(y)|$$

$$\le t(1-t)\mathrm{lip}(f)\|x-y\| + t(1-t)\mathrm{lip}(f)\|x-y\| = 2t(1-t)\mathrm{lip}(f)\|x-y\|.$$

□

We denote

$$d_n(x) := \operatorname{dist}(x; \frac{1}{2^{n-1}}\mathbb{Z}) \quad \text{for } n \in \mathbb{N}, x \in \mathbb{R}.$$

It is obvious that d_n is periodic with period $1/2^{n-1}$.

Lemma 2.1. *Let $n \in \mathbb{N}$. Then*

$$d_n(x) = \begin{cases} |x - \frac{k}{2^n}| \text{ if } k \in 2\mathbb{Z}, \\ \frac{1}{2^n} - |x - \frac{k}{2^n}| \text{ if } k \in 2\mathbb{Z}+1, \end{cases} \qquad \text{for } x \in [\tfrac{k-1}{2^n}, \tfrac{k+1}{2^n}].$$

Proof. Since d_n is periodic with period $1/2^n$, it is enough to consider the case when $k = 0$ or $k = 1$.

If $k = 0$, then for $x \in [-\frac{1}{2^n}, \frac{1}{2^n}]$ we have

$$d_n(x) = \operatorname{dist}(x; \tfrac{1}{2^{n-1}}\mathbb{Z}) = \operatorname{dist}(x; \{0\}) = |x|.$$

If $k = 1$, then for $x \in [0, \frac{2}{2^n}]$ we have

$$d_n(x) = \operatorname{dist}(x; \tfrac{1}{2^{n-1}}\mathbb{Z}) = \operatorname{dist}(x; \{0, \tfrac{2}{2^n}\}) = \frac{1}{2^n} - |x - \frac{1}{2^n}|.$$

□

For a sequence $a = (a_i)_{i\in\mathbb{N}} \subset \mathbb{R}_+$ and $k \in \mathbb{N}$, $l \in \mathbb{N} \cup \{\infty\}$, $l \geq k$ we consider the function $T_k^l[a] : \mathbb{R} \to [0, \infty]$ defined by

$$T_k^l[a](x) := \sum_{i=k}^{l} a_i d_i(x) \quad \text{for } x \in \mathbb{R}.$$

Instead of $T_1^\infty[a]$ we write $T[a]$. Clearly $T_k^l[a]$ is periodic with period $1/2^k$. We use the convention that $\sum_{i=1}^{0} = 0$, which implies that $T_1^0[a] = 0$.

Lemma 2.2. *Let $a = (a_i)_{i\in\mathbb{N}} \subset \mathbb{R}_+$ be given, and let $k \in \mathbb{Z}$, $n \in \mathbb{N}$ be fixed. If $n = 1$ or $k \in 2\mathbb{Z}+1$ then $T_1^{n-1}[a]$ is affine on $[\frac{k-1}{2^n}, \frac{k+1}{2^n}]$.*

Proof. If $n = 1$, then $T_1^{n-1}[a] = T_1^0[a] = 0$, which trivially yields the assertion.

Consider now the case when $n \geq 2$. Then $k = 2m+1$ for a certain $m \in \mathbb{Z}$.

Since the sum of affine functions is affine, it is enough to show that d_i is affine on

$$[\frac{k-1}{2^n}, \frac{k+1}{2^n}] = [\frac{m}{2^{n-1}}, \frac{m+1}{2^{n-1}}],$$

for every $i \in \{1, \ldots, n-1\}$.

For each $i \in \{1, \ldots, n-1\}$ there exists $m_i \in \mathbb{Z}$ such that

$$[\frac{m}{2^{n-1}}, \frac{m+1}{2^{n-1}}] \subset [\frac{m_i}{2^i}, \frac{m_i+1}{2^i}].$$

If $m_i \in 2\mathbb{Z}$, then by Lemma 2.1

$$d_i(x) = |x - \frac{m_i}{2^i}| = x - \frac{m_i}{2^i} \quad \text{for } x \in [\frac{m_i}{2^i}, \frac{m_i+1}{2^i}],$$

while if $m_i \in 2\mathbb{Z}+1$ we get

$$d_i(x) = \frac{1}{2^i} - |x - \frac{m_i}{2^i}| = \frac{m_i+1}{2^i} - x \quad \text{for } x \in [\frac{m_i}{2^i}, \frac{m_i+1}{2^i}].$$

□

Lemma 2.3. *Let $a = (a_i)_{i\in\mathbb{N}} \subset \mathbb{R}_+$ be given and let $k \in \mathbb{Z}$, $l, n \in \mathbb{N}$, $l \geq n$ be fixed. Then for $x \in [\frac{k}{2^n} - \frac{1}{2^l}, \frac{k}{2^n} + \frac{1}{2^l}]$*

$$T_n^l[a](x) = \begin{cases} (a_n + \ldots + a_l)|x - \frac{k}{2^n}| \text{ if } k \in 2\mathbb{Z}, \\ \frac{a_n}{2^n} + ((-a_n) + a_{n+1} + \ldots + a_l)|x - \frac{k}{2^n}| \text{ if } k \in 2\mathbb{Z}+1. \end{cases}$$

Proof. Consider an arbitrary $x \in [\frac{k}{2^n} - \frac{1}{2^l}, \frac{k}{2^n} + \frac{1}{2^l}]$. We have

$$[\frac{k}{2^n} - \frac{1}{2^l}, \frac{k}{2^n} + \frac{1}{2^l}] \subset [\frac{2^{i-n}k}{2^i} - \frac{1}{2^i}, \frac{2^{i-n}k}{2^i} + \frac{1}{2^i}] \quad \text{for } i = n, \ldots, l. \tag{4}$$

If $k \in 2\mathbb{Z}$ then by (4) and Lemma 2.1 we obtain that

$$d_i(x) = |x - \frac{2^{i-n}k}{2^i}| = |x - \frac{k}{2^n}| \quad \text{for } i = n, \ldots, l,$$

and consequently

$$T_n^l[a](x) = \sum_{i=n}^{l} a_i d_i(x) = (\sum_{i=n}^{l} a_i)|x - \frac{k}{2^n}|.$$

Assume now that $k \in 2\mathbb{Z}+1$. Making use of (4) for $i = n$ and Lemma 2.1 we get

$$d_n(x) = \frac{1}{2^n} - |x - \frac{k}{2^n}|.$$

Since $2^{i-n}k \in 2\mathbb{Z}$ for $i = n+1, \ldots, l$, by (4) and Lemma 2.1,

$$d_i(x) = |x - \frac{k}{2^n}| \quad \text{for } i = n+1, \ldots, l.$$

Thus

$$T_n^l[a](x) = a_n d_n(x) + \sum_{i=n+1}^{l} a_i d_i(x) = \frac{a_n}{2^n} + ((-a_n) + a_{n+1} + \ldots + a_l)|x - \frac{k}{2^n}|.$$

□

Lemma 2.4. *For $n \in \mathbb{N}$ we have*

a) $\mathcal{C}d_n(x, y; t) \leq 2t(1-t)|x - y|$ for $x, y \in \mathbb{R}$, $t \in [0, 1]$;

b) $\mathcal{C}d_n(x, y; t) \in [-\frac{1}{2^n}, \frac{1}{2^n}]$ for $x, y \in \mathbb{R}$, $t \in [0, 1]$.

Proof. It is clear that d_n is Lipschitz with $\mathrm{lip}(d_n) = 1$. By Proposition 2.1 we get a). By the definition of the operator $\mathcal{C}$ we have for $x, y \in \mathbb{R}$, $t \in [0, 1]$

$$\begin{array}{rcl} \mathcal{C}d_n(x, y; t) & = & d_n(tx + (1-t)y) - td_n(x) - (1-t)d_n(y) \\ & \in & [0, \frac{1}{2^n}] - [0, \frac{1}{2^n}] = [-\frac{1}{2^n}, \frac{1}{2^n}]. \end{array}$$

□

Lemma 2.5. *Let $a = (a_i)_{i \in \mathbb{N}} \subset \mathbb{R}_+$ be a given sequence. We assume that there exists a $q > 1/2$ such that*

$$a_{i+1} \geq q a_i \quad \textit{for } i \in \mathbb{N}.$$

Let $K_q \in \mathbb{N}$ be such that

$$q + \ldots + q^{K_q} > 1. \tag{5}$$

Let $n \in \mathbb{N}$ and $l \in \mathbb{N}$, $l \geq n + K_q$ be arbitrary. Then $T_n^l[a]$ is convex on $[\frac{k}{2^n} - \frac{1}{2^l}, \frac{k}{2^n} + \frac{1}{2^l}]$ for every $k \in \mathbb{Z}$.

Proof. We have

$$a_{n+1} + \ldots + a_l \geq a_n(q + \ldots + q^{K_q}) \geq a_n,$$

and hence

$$(-a_n) + a_{n+1} + \ldots + a_l \geq 0.$$

Lemma 2.3 completes the proof. □

Lemma 2.6. *Let $x, y \in \mathbb{R}$, $x < y < x + 1/2$. Let n be the smallest positive integer such that*

$$(x, y) \cap \tfrac{1}{2^n}\mathbb{Z} \neq \emptyset.$$

Then the following statements hold:

i) There exists a unique $k \in \mathbb{Z}$ such that $\frac{k}{2^n} \in (x, y)$. Moreover, if $n > 1$ then $k \in 2\mathbb{Z} + 1$.

ii) There exists the greatest $l \in \mathbb{N}$ such that

$$[x, y] \subset [\frac{k}{2^n} - \frac{1}{2^l}, \frac{k}{2^n} + \frac{1}{2^l}]. \tag{6}$$

Moreover, then $l \geq n$ and

$$\frac{1}{4}\frac{1}{2^l} \leq y - x \leq 2\frac{1}{2^l}. \tag{7}$$

Proof. i) The existence of $k \in \mathbb{Z}$ such that $\frac{k}{2^n} \in (x, y)$ follows from the definition of n. To prove its uniqueness suppose that there exist $k_1, k_2 \in \mathbb{Z}$, $k_1 < k_2$ such that $\frac{k_1}{2^n}, \frac{k_2}{2^n} \in (x, y)$. Then $\frac{k_1}{2^n}, \frac{k_1+1}{2^n} \in (x, y)$. One of the numbers k_1, k_1+1 is even. Suppose e.g. that $k_1 \in 2\mathbb{Z}$. Then $\frac{k_1}{2} \in \mathbb{Z}$ and $\frac{k_1/2}{2^{n-1}} = \frac{k_1}{2^n} \in \mathbb{Z}$, which contradicts the definition of n.

Now we prove the second part of i). Suppose that $n > 1$ and that there exists a $p \in \mathbb{Z}$ such that $\frac{2p}{2^n} \in (x, y)$. Then we would get $\frac{p}{2^{n-1}} = (2p)/2^n \in (x, y)$, and since $n - 1 \in \mathbb{N}$ we again obtain a contradiction.

ii) We first prove that $l = n$ satisfies (6). We have to show that $\frac{k-1}{2^n} \leq x$ and that $\frac{k+1}{2^n} \geq y$. Suppose for an indirect proof, that either $\frac{k-1}{2^n} > x$ or $\frac{k+1}{2^n} < y$. We consider the case $\frac{k-1}{2^n} > x$. Obviously $\frac{k-1}{2^n} < y$. Hence $\frac{k-1}{2^n} \in (x, y)$, which contradicts i). The reasoning in the case $\frac{k+1}{2^n} < y$ is analogous.

For sufficiently large $l \in \mathbb{N}$ we have

$$(\frac{k}{2^n} - \frac{1}{2^l}, \frac{k}{2^n} + \frac{1}{2^l}) \subset (x, y).$$

It means that the set of integers l satisfying (6) is bounded above. Therefore there exists the great l in this set. It remains to prove that it satisfies (7). From (6) we get

$$y - x \leq (\frac{k}{2^n} + \frac{1}{2^l}) - (\frac{k}{2^n} - \frac{1}{2^l}) = \frac{2}{2^l}.$$

Now we prove that $y - x \geq \frac{1}{4}\frac{1}{2^l}$. Suppose that it is not the true, that is $y - x < \frac{1}{4 \cdot 2^l}$. Since l is the greatest integer satisfying (6), either $x < \frac{k}{2^n} - \frac{1}{2 \cdot 2^l}$ or $y > \frac{k}{2^n} + \frac{1}{2 \cdot 2^l}$. If $x < \frac{k}{2^n} - \frac{1}{2 \cdot 2^l}$ then we would get

$$y = y - x + x < \frac{1}{4 \cdot 2^l} + \frac{k}{2^n} - \frac{1}{2 \cdot 2^l} < \frac{k}{2^n},$$

a contradiction. Similarly, if $y > k/2^n + 1/(2 \cdot 2^l)$, we would get

$$x = y + (-y + x) > \frac{k}{2^n} + \frac{1}{2 \cdot 2^l} - \frac{1}{4 \cdot 2^l} > \frac{k}{2^n},$$

a contradiction. $\square$

3 Paraconvexity

In this section we investigate the problem when the function $T[a]$ is paraconvex.

Theorem 3.1. *Let $a = (a_i)_{i \in \mathbb{N}} \subset \mathbb{R}_+$ be such that $T[a]$ is paraconvex. Then*

$$\lim_{n \to \infty} a_n = 0.$$

Proof. We have

$$a_n = \mathcal{C}T[a](0, 2^{-n-1}; \tfrac{1}{2})/(2^{-n}) \leq \frac{\gamma(2^{-(n-1)})}{2^{-n}} \to 0 \text{ as } n \to \infty.$$

$\square$

It occurs that the condition $\lim_{n \to \infty} a_n = 0$ does not guarantee even the local paraconvexity of the function $T[a]$.

Theorem 3.2. *Let U be a nonempty open subinterval of $\mathbb{R}$ and let $a = (a_i)_{i \in \mathbb{N}} \subset (0, \infty)$. We assume that there exists a $q < 1/2$ such that*

$$\frac{a_{i+1}}{a_i} \leq q \quad \text{for } i \in \mathbb{N}.$$

Then $T[a]|_U$ is not paraconvex.

Proof. We can find $n \in \mathbb{N}, n \geq 1$ and $k \in 2\mathbb{Z}+1$ such that

$$[\frac{k-1}{2^n}, \frac{k+1}{2^n}] \subset U.$$

Fix arbitrarily an $l \in \mathbb{N}$, $l \geq n$. By Lemma 2.3 we have $T_n^\infty[a](\frac{k}{2^n}) = \frac{a_n}{2^n}$ and

$$T_n^l[a](\frac{k}{2^n}+\frac{1}{2^l}) = T_n^l[a](\frac{k}{2^n}-\frac{1}{2^l}) = a_n(\frac{1}{2^n}-\frac{1}{2^l}) + \sum_{i=n+1}^{l-1}\frac{1}{2^l}a_i$$

$$\leq a_n(\frac{1}{2^n}-\frac{1}{2^l}) + \frac{1}{2^l}\sum_{i=n+1}^{l-1} a_n q^{i-n} = a_n(\frac{1}{2^n}-\frac{1}{2^l}) + \frac{1}{2^l}a_n q\frac{1-q^{l-n-1}}{1-q}.$$

Since $d_i(\frac{k}{2^n} \pm \frac{1}{2^l}) = 0$ for $i > l$, we obtain that

$$T_n^\infty[a](\frac{k}{2^n} \pm \frac{1}{2^l}) \leq a_n(\frac{1}{2^n}-\frac{1}{2^l}) + \frac{1}{2^l}a_n q\frac{1-q^{l-n-1}}{1-q}.$$

By Lemma 2.2 $T_1^{n-1}[a]$ is affine on the interval $[\frac{k}{2^n}-\frac{1}{2^l}, \frac{k}{2^n}+\frac{1}{2^l}]$ and therefore $\mathcal{C}T_1^{n-1}[a]|_{[\frac{k}{2^n}-\frac{1}{2^l}, \frac{k}{2^n}+\frac{1}{2^l}]} = 0$. Whence by the above estimation we get

$$\mathcal{C}T[a](\frac{k}{2^n}-\frac{1}{2^l}, \frac{k}{2^n}+\frac{1}{2^l}; \frac{1}{2}) = \mathcal{C}T_n^\infty[a](\frac{k}{2^n}-\frac{1}{2^l}, \frac{k}{2^n}+\frac{1}{2^l}; \frac{1}{2})$$

$$\geq \frac{1}{2^l}(a_n - a_n q\frac{1-q^{l-n-1}}{1-q}),$$

and consequently

$$\mathcal{C}T[a](\frac{k}{2^n}-\frac{1}{2^l}, \frac{k}{2^n}+\frac{1}{2^l}; \frac{1}{2})/|(\frac{k}{2^n}+\frac{1}{2^l}) - (\frac{k}{2^n}-\frac{1}{2^l})|$$

$$\geq 2(a_n - a_n q\frac{1-q^{l-n-1}}{1-q}) \to 2a_n\frac{1-2q}{1-q} > 0 \text{ as } l \to \infty.$$

It proves that $T[a]|_U$ is not paraconvex. □

Theorem 3.3. *Let $a = (a_i)_{i\in\mathbb{N}} \subset (0,\infty)$ be a sequence such that $\lim_{n\to\infty} a_n = 0$. We assume that there exists a $q > 1/2$ satisfying*

$$\frac{a_{i+1}}{a_i} \geq q \quad \text{for } i \in \mathbb{N}.$$

Then $T[a]$ is paraconvex.

Proof. Let K_q be the number satisfying (5). We define a function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$. We put $\omega(0) = 0$,

$$\omega(r) := \max\{a_i \,|\, i \in \mathbb{N}, i \geq -\log_2 r - K_q - 1\} \quad \text{for } r > 0.$$

It is clear that ω is nondecreasing and that $\lim_{r\to 0^+} \omega(r) = 0$. We will show that

$$\mathcal{C}T[a](x, y; t) \leq 2^{K_q+2}|x - y|\omega(|x - y|) \quad \text{for } x, y \in \mathbb{R}, t \in [0, 1]. \tag{8}$$

Consider arbitrary $x, y \in \mathbb{R}$, $x < y$, $|x - y| < 1/2$ and arbitrary $t \in [0, 1]$. Let n, k, l be as in Lemma 2.6. By Lemmas 2.6 and 2.2 we obtain that $T_1^{n-1}[a]$ is affine on $[x, y]$. Therefore we have

$$\mathcal{C}T[a](x, y; t) = \mathcal{C}T_n^\infty(x, y; t) \quad \text{for } t \in [0, 1].$$

Two cases may occur:

a) $l < n + K_q$,

b) $l \geq n + K_q$.

Consider first the case a). Then $n + K_q + 1 \geq l + 2$ and by (7) we get

$$\frac{1}{2^{n+K_q+1}} \leq \frac{1}{2^{l+2}} \leq |x - y|.$$

This yields that

$$n \geq -\log_2 |x - y| - K_q - 1,$$

or equivalently

$$2^{-n} \leq 2^{K_q+1}|x - y|.$$

Making use of the last two inequalities, Lemma 2.4 b) and definition of ω we obtain

$$\begin{aligned}
\mathcal{C}T_n^\infty[a](x, y; t) &= \sum_{i=n}^{\infty} \mathcal{C}d_i(x, y; t) \\
&\leq \max\{a_i \,|\, i \geq n\} \sum_{i=n}^{\infty} \mathcal{C}d_i(x, y; t) \\
&\leq \max\{a_i \,|\, i \geq -\log_2 |x - y| - K_q - 1\} \textstyle\sum_{i=n}^{\infty} \frac{1}{2^i} \\
&= \omega(|x - y|)\tfrac{1}{2^{n-1}} \leq 2^{K_q+2}|x - y|\omega(|x - y|).
\end{aligned}$$

We have proved (8).

Now we consider the case b). It follows from Lemma 2.3 that $T_n^l[a]$ is convex on the interval $[\frac{k}{2^n} - \frac{1}{2^l}, \frac{k}{2^n} + \frac{1}{2^l}]$. Hence, by (6) $T_n^l[a]$ is convex on $[x, y]$. Whence it follows that

$$\mathcal{C}T_n^\infty[a](x, y; t) \leq \mathcal{C}T_{l+1}^\infty[a](x, y; t).$$

By (7) we have

$$2^{-l-2} \leq |x - y|,$$

and consequently

$$l + 1 \geq -\log_2 |x - y| - 1.$$

From the above inequality, Lemma 2.4 b) and (7) we get

$$\begin{aligned}
\mathcal{C}T_{l+1}^\infty[a](x, y; t) &= \sum_{i=l+1}^{\infty} a_i \mathcal{C}d_i(x, y; t) \\
&\leq \max\{a_i \,|\, i \geq l + 1\} \sum_{i=l+1}^{\infty} \tfrac{1}{2^i} \\
&\leq \max\{a_i \,|\, i \geq -\log_2 |x - y| - 1\} \tfrac{1}{2^l} \\
&\leq \omega(|x - y|) \tfrac{1}{2^l} \leq \omega(|x - y|) 4|x - y| \\
&\leq 2^{K_q+2} |x - y| \omega(|x - y|).
\end{aligned}$$

We have proved (8)

Consider now the case when $|x - y| \geq 1/2$. By Lemma 2.4 b) we obtain

$$\mathcal{C}T[a](x, y; t) = \sum_{i=1}^{\infty} a_i \mathcal{C}d_i(x, y; t) \leq \sum_{i=1}^{\infty} \frac{a_i}{2^i}.$$

We are going to compare the above estimation with (8). We denote

$$\gamma(r) := 2^{K_q+2} r \omega(r) \quad \text{for } r \in \mathbb{R}_+. \tag{9}$$

We have

$$\omega(\tfrac{1}{2}) = \max\{a_i \,|\, i \geq -\log_2(\tfrac{1}{2}) - K_q - 1\} = \max\{a_i \,|\, i \in \mathbb{N}\},$$

and consequently

$$\gamma(\tfrac{1}{2}) = 2^{K_q+1} \max\{a_i \,|\, i \in \mathbb{N}\} \geq \max\{a_i \,|\, i \in \mathbb{N}\} \sum_{i=1}^{\infty} \frac{1}{2^i} \geq \sum_{i=1}^{\infty} \frac{a_i}{2^i}.$$

Hence (8) is valid also in the case when $|x - y| \geq 1/2$.

We have proved (8), which means that the function $T[a]$ is paraconvex with γ defined by (9). $\square$

Remark 3.1. Let us observe that the case $q = 1/2$ is in a sense boundary point. In Theorem 3.2 we have shown that if $a = (a_i) \subset (0, \infty)$ and

$$\frac{a_{i+1}}{a_i} \le q < \frac{1}{2} \quad \text{for } i \in \mathbb{N},$$

then $T[a]$ is not paraconvex. In the case when $a = (\frac{1}{2^i})_{i\in\mathbb{N}}$ we have (see for example [7, 15])

$$T[(2^{-i})_{i\in\mathbb{N}}](x) = 2x(1-x) \quad \text{for } x \in [0,1].$$

Whence we immediately obtain that if $a = (a_i)_{i\in\mathbb{N}} \subset (0, \infty)$ and

$$\frac{a_{i+1}}{a_i} = \frac{1}{2} \quad \text{for } i \in \mathbb{N},$$

then

$$T[a](x) = 4a_1 x(1-x) \quad \text{for } x \in [0,1].$$

One can easily check that this function is paraconvex with $\gamma(r) = 2a_1 r^2$. By Theorem 3.3 if $\lim\limits_{i\to\infty} a_i = 0$ and there exists q such that

$$\frac{a_{i+1}}{a_i} \ge q > \frac{1}{2} \quad \text{for } i \in \mathbb{N},$$

then $T[a]$ is paraconvex.

To show an important consequence of Theorem 3.3 we need the result of Kôno.

Theorem of Kôno [8, Theorem 2]. *Let $a = (a_i)_{i\in\mathbb{N}} \subset \mathbb{R}$ be a sequence such that $\sum\limits_{i=1}^{\infty} 2^i |a_i| < \infty$. Then*

(i) $T[a]$ is absolutely continuous if and only if $\sum\limits_{i=1}^{\infty} a_i^2 < \infty$.

(ii) $T[a]$ is differentiable on a set of cardinality continuum and the range of the derivative is a whole line but there exists no derivative almost surely if and only if $\lim\limits_{i\to\infty} a_i = 0$ but $\sum\limits_{i=1}^{\infty} a_i^2 = \infty$.

(iii) $T[a]$ has nowhere finite derivative if and only if $\liminf\limits_{i\to\infty} |a_i| > 0$.

Directly from Theorem 3.3 and Theorem of Kôno we get:

Corollary 3.1. *Let $a = (a_i)_{i\in\mathbb{N}} \subset (0,\infty)$ be a sequence such that*

- $\lim\limits_{n\to\infty} a_n = 0$;
- $\sum\limits_{i=1}^{\infty} a_i^2 = \infty$;
- *there exists a $q > 1/2$ such that $a_{i+1} \geq q a_i$ for $i \in \mathbb{N}$;*

Then $T[a]$ is paraconvex function which is differentiable only on a set of measure zero.

Clearly $a_n = \frac{1}{\sqrt{n}}$ is a sequence satisfying the assumptions of the above corollary, which implies that the function

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \operatorname{dist}(x; \tfrac{1}{2^{n-1}}\mathbb{Z})$$

is an example of a paraconvex function which is differentiable only on a set of measure zero (and consequently is not strongly paraconvex).

4 Strong paraconvexity

As we know Takagi class functions are usually very irregular. In this section we will investigate the question when the elements of Takagi class are strongly paraconvex. We begin with:

Theorem 4.1. *Let $a = (a_i)_{i\in\mathbb{N}} \subset \mathbb{R}_+$ be a sequence such that $T[a]$ is strongly paraconvex. Then*

$$\sum_{i=1}^{\infty} a_i < \infty.$$

Proof. We have

$$\frac{1}{2^n}\sum_{i=1}^{n} a_i = T[a](\frac{1}{2^n}) = \mathcal{C}T[a](1,0;\frac{1}{2^n}) \leq \min(\frac{1}{2^n}, 1-\frac{1}{2^n})\gamma(1) \leq \frac{1}{2^n}\gamma(1).$$

Whence we immediately obtain the assertion. □

Now we prove a sufficient condition. The idea of the proof is similar to that of Theorem 3.3.

Theorem 4.2. *Let $a = (a_i)_{i\in\mathbb{N}} \subset \mathbb{R}_+$ be a sequence such that there exists a $q > 1/2$ satisfying*

$$\frac{a_{i+1}}{a_i} \geq q \quad \text{for } i \in \mathbb{N}.$$

If

$$\sum_{i=1}^{\infty} a_i < \infty, \tag{10}$$

then the function $T[a]$ is strongly paraconvex.

Proof. Let K_q be the constant satisfying (5). We define the function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$. We put $\omega(0) = 0$,

$$\omega(r) := \sum_{\substack{i\in\mathbb{N},\\ i\geq -\log_2 r - K_q - 1}} a_i \quad \text{for } r > 0.$$

It is clear that ω is nondecreasing. It follows from (10) that $\lim_{r\to 0} \omega(r) = 0$. We will show that

$$\mathcal{C}T[a](x, y; t) \leq 2t(1-t)|x-y|\omega(|x-y|) \quad \text{for } x, y \in \mathbb{R}, t \in [0, 1]. \tag{11}$$

Let $x, y \in \mathbb{R}$, $x < y$, $t \in [0, 1]$. We consider first the case when $|x - y| \leq 1/2$. Let n, k, l be as in Lemma 2.6. By the same argumentation as in the proof of Theorem 3.3 we obtain that

$$\mathcal{C}T[a](x, y; t) = \mathcal{C}T_n^{\infty}[a](x, y; t).$$

Again proceeding as in that proof we consider two cases:

a) $l \leq n + K_q$,

b) $l \geq n + K_q$.

In the first case we get that

$$n \geq -\log_2|x-y| - K_q - 1,$$

which means that

$$\sum_{i=n}^{\infty} a_i \leq \omega(|x-y|).$$

Making use of Lemma 2.4 a) we obtain

$$\begin{aligned}&\mathcal{C}T_n^\infty[a](x,y;t) = \textstyle\sum_{i=n}^\infty a_i \mathcal{C}d_i(x,y;t)\\ &\le (\textstyle\sum_{i=n}^\infty a_i)\cdot \max_{i\in\mathbb{N},i\ge n} \mathcal{C}d_i(x,y;t)\\ &\le \omega(|x-y|)2t(1-t)|x-y|.\end{aligned}$$

We have proved (11) in the case a).

Now we consider the case b). As in the proof of Theorem 3.3 we obtain

$$\mathcal{C}T_n^\infty[a](x,y;t) \le \mathcal{C}T_{l+1}^\infty[a](x,y;t)$$

and

$$l+1 \ge -\log_2|x-y|-1.$$

Applying the above inequalities, definition of ω and Lemma 2.4 i) we get

$$\begin{aligned}&\mathcal{C}T[a]_n^\infty(x,y;t) \le \mathcal{C}T_{l+1}^\infty[a](x,y;t)\\ &\le (\textstyle\sum_{i=l+1}^\infty a_i)\max_{i\in\mathbb{N},i\ge l+1}\mathcal{C}d_i(x,y;t)\\ &\le \omega(|x-y|)2t(1-t)|x-y|.\end{aligned}$$

We have proved (11). It remains to consider the case when $|x-y| \ge 1/2$. By Lemma 2.4 i) we have

$$\mathcal{C}T[a](x,y;t) = \sum_{i=1}^\infty a_i\mathcal{C}d_i(x,y;t) \le (\sum_{i=1}^\infty a_i)2t(1-t)|x-y|.$$

On the other hand for $r \ge 1/2$ we have

$$\omega(r) = \sum_{i\in\mathbb{N},i\ge -\log_2 r-K_q-1} a_i = \sum_{i=1}^\infty a_i.$$

Thus in the considered case we have

$$\mathcal{C}T[a](x,y;t) \le 2t(1-t)|x-y|\omega(|x-y|),$$

which means that (11) is valid, an consequently $T[a]$ is γ-strongly paraconvex, with $\gamma(r) := 2r\omega(r)$. □

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