When are maps preserving semi-inner products linear?

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Let $(X, \|\cdot\|)$ be a normed space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. By S_X we denote the unit sphere in X. Let X^* be the collection of all continuous linear functionals on X. Lumer [2] proved that there always exists a mapping $[\cdot|\diamond] : X \times X \to \mathbb{K}$ satisfying the following properties:

- $\forall_{x,y,z \in X}\ \forall_{\alpha,\beta \in \mathbb{K}} : \quad [\alpha x + \beta y|z] = \alpha\,[x|z] + \beta\,[y|z];$
- $\forall_{x,y \in X}\ \forall_{\alpha \in \mathbb{K}} : \quad [x|\alpha y] = \overline{\alpha}\,[x|y];$
- $\forall_{x,y \in X} : \quad |\,[x|y]\,| \leqslant \|x\| \cdot \|y\|, \quad [x|x] = \|x\|^2.$

Such a mapping is called a *semi–inner product* in X.

[2] G. Lumer, *Semi-inner-product spaces*, Trans. Am. Math. Soc., **100** (1961), 29–43.

A *supporting functional* $\varphi_x \colon X \to \mathbb{K}$ *at* $x \in X$ is a norm-one linear functional in X^* such that $\varphi_x(x) = \|x\| = \langle \varphi_x, x \rangle$. By the Hahn-Banach theorem there always exists at least one such functional for every $x \in X$. There may exist infinitely many different semi-inner products in X. There is a unique one if and only if X is *smooth* (i.e. there is a unique supporting functional at each point of the set $X \setminus \{0\}$). Then

$$[x|y] = \|y\| \cdot \varphi_y(x) = \|y\| \cdot \langle \varphi_y, x \rangle \quad \text{for all} \quad x, y \in X. \tag{1}$$

The following result appeared in

[1] D. Ilišević, A. Turnšek, *On Wigner's theorem in smooth normed spaces*, Aequationes Math. **94** (2020), 1257–1267.

Proposition

[1, Proposition 2.4] *Let* X, Y *be normed spaces and* $f\colon X \to Y$ *a mapping such that* $[f(x)|f(y)] = [x|y]$, $x, y \in X$.
(i) *If* f *is surjective, then* f *is a linear isometry.*
(ii) *If* $X = Y$ *is a smooth Banach space, then* f *is a linear isometry.*

The above property (ii) is not true.
Unluckily, the proof of [1, Proposition 2.4] contains a small flaw.

In the proof of [1, Proposition 2.4], the authors postulated the following inclusion:

$$\{\xi\varphi_{f(z)} \in X^*\colon z \in X,\, \xi \in \mathbb{C}\} \supseteq \{\xi\varphi_{f(z)} \circ f \in X^*\colon z \in X,\, \xi \in \mathbb{C}\}, \quad (2)$$

where $\varphi_{f(z)} \circ f = \varphi_z$ (see [1, p. 1265, third line from the bottom]), which fails already in the Hilbert-space setting.

To see this, let us consider the Hilbert space ℓ_2. The only semi-inner product on ℓ_2 is the inner product $\langle\cdot|\cdot\rangle$ itself. We consider the unilateral shift on ℓ_2, which is a non-surjective isometry $f\colon \ell_2 \to \ell_2$; it is given by the formula $f(x) = f(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, x_3, \ldots)$. It is easy to see that $\langle f(x)|f(y)\rangle = \langle x|y\rangle$ and

$$\Big\{\xi\varphi_{f(z)} \circ f \in (\ell_2)^*\colon z \in \ell_2,\, \xi \in \mathbb{C}\Big\} = (\ell_2)^*.$$

On the other hand,

$$\Big\{\xi\varphi_{f(z)} \in (\ell_2)^*\colon z \in \ell_2,\, \xi \in \mathbb{C}\Big\} = \Big\{\langle\cdot|w\rangle \in (\ell_2)^*\colon w = (0, w_1, w_2, \ldots) \in \ell_2\Big\},$$

which demonstrates that indeed (2) is fatally flawed.

Theorem

Let X and Y be normed spaces with fixed semi-inner products $[\cdot|\cdot]_X$ and $[\cdot|\cdot]_Y$, respectively. Suppose that $f: X \to Y$ is a function such that

$$[f(x)|f(y)]_Y = [x|y]_X \qquad (x, y \in X). \tag{3}$$

(i) *If* $\dim X = \dim Y = n < \infty$, *then f is a linear isometry.*
(ii) *If X has a Schauder basis (e_i) and $(f(e_i))$ is a Schauder basis of Y, then f is a linear isometry.*

Proof: We will prove clause (ii) first. Suppose that (e_i) is a Schauder basis of X and $(f(e_i))$ is a Schauder basis of Y. We will show that for any scalars $\beta_1, \beta_2, \ldots$

$$f\left(\sum_{i=1}^{\infty} \beta_i e_i\right) = \sum_{i=1}^{\infty} \beta_i f(e_i)$$

as long as the series $\sum_{i=1}^{\infty} \beta_i e_i$ converges in X.

Fix $x \in X$. Since $(f(e_i))_{i=1}^{\infty}$ is a basis, there are uniquely determined scalars $\beta_1, \beta_2, \ldots$ such that

$$f(x) = \sum_{i=1}^{\infty} \beta_i f(e_i).$$

Let $x_m := \sum_{i=1}^{m} \beta_i e_i$. It is enough to show that $x_m \to x$ as $m \to \infty$. Let us define the numbers $\varepsilon_m := \|f(x) - \sum_{i=1}^{m} \beta_i f(e_i)\|$. Clearly, $\varepsilon_m \to 0$ as $m \to \infty$. For every unit vector $u \in X$, we have $1 = \|u\| = \|f(u)\|$. Thus

$$\left| \left[f(x) - \sum_{i=1}^{m} \beta_i f(e_i) | f(u) \right] \right| \leqslant \left\| f(x) - \sum_{i=1}^{m} \beta_i f(e_i) \right\| \cdot \|f(u)\| \leqslant \varepsilon_m \cdot 1.$$

Using linearity of the semi-inner products in the first variable, we get

$$\left| [f(x)|f(u)] - \left[\sum_{i=1}^{m} \beta_i f(e_i) | f(u) \right] \right| = \left| [f(x)|f(u)] - \sum_{i=1}^{m} \beta_i \left[f(e_i) | f(u) \right] \right| \leqslant \varepsilon_m.$$

Combining the above inequality with (3) yields $\left| [x|u] - \sum_{i=1}^{m} \beta_i \left[e_i | u \right] \right| \leqslant \varepsilon_m.$

Consequently, $\left| \left[x - \sum_{i=1}^{m} \beta_i e_i | u \right] \right| \leqslant \varepsilon_m$, which means that
$\left| \, [x - x_m | u] \, \right| \leqslant \varepsilon_m$.
It is known that $\{ \, [\, \cdot \, | w] \in X^* \colon \|w\| = 1, w \in X \}$ is a 1-norming subset in the dual ball of X^*, i.e.

$$\|a\| = \sup \{ \, [a|w] \in X^* \colon \|w\| = 1, w \in X \}.$$

This allows us to conclude that we have $\|x - x_m\| \leqslant \varepsilon_m$, so $x_m \to x$. We have thus proved that $x = \sum_{i=1}^{\infty} \beta_i e_i$.
It is helpful to recall: $f(x) = \sum_{i=1}^{\infty} \beta_i f(e_i)$. So, we have

$$f \left(\sum_{i=1}^{\infty} \beta_i e_i \right) = \sum_{i=1}^{\infty} \beta_i f(e_i).$$

In particular, f is linear, hence also isometric because it preserves the semi-inner products.

Now, in order to prove clause (i), it is enough to show that f maps linearly independent sets to linearly independent sets.

Let $n = \dim X$. Fix a basis $\{b_1, \ldots, b_n\}$ for X. We *claim* that the set $\{f(b_1), \ldots, f(b_n)\}$ is linearly independent in Y. To see this, suppose that $\sum_{k=1}^{n} \alpha_k f(b_k) = 0$. It follows from (3) that

$$\begin{aligned}
\left\| \sum_{k=1}^{n} \alpha_k b_k \right\|^2 &= \left[\sum_{k=1}^{n} \alpha_k b_k \Big| \sum_{k=1}^{n} \alpha_k b_k \right] \\
&= \sum_{k=1}^{n} \alpha_k \left[b_k \Big| \sum_{k=1}^{n} \alpha_k b_k \right] = \sum_{k=1}^{n} \alpha_k \left[f(b_k) \Big| f\left(\sum_{k=1}^{n} \alpha_k b_k \right) \right] \\
&= \left[\sum_{k=1}^{n} \alpha_k f(b_k) \Big| f\left(\sum_{k=1}^{n} \alpha_k b_k \right) \right] = \left[0 \Big| f\left(\sum_{k=1}^{n} \alpha_k b_k \right) \right] = 0.
\end{aligned}$$

Hence $\sum_{k=1}^{n} \alpha_k b_k = 0$. Since the vectors $b_1, \ldots, b_n$ are linearly independent, we have $\alpha_1 = \ldots = \alpha_n = 0$. This means that the vectors $f(b_1), \ldots, f(b_n)$ are linearly independent too. Consequently, $\{f(b_1), \ldots, f(b_n)\}$ is a basis for Y. Now, we can apply (ii). ■

Remark: It is worth mentioning that for finite-dimensional case, the assumption $\dim X = \dim Y$ is the best that can be said. Namely, an inequality $\dim X \leqslant \dim Y$ does not imply linearity of f; see the paper:

[3] P. Wójcik, *On an orthogonality equation in normed spaces*, Funct. Anal. Appl. **52**(3) (2018) 224–227.

Indeed, it was observed in [3] that if X is a non-Hilbertian finite-dimensional space with $n = \dim X \geqslant 3$ that is smooth, then there exists a space V of dimension $n - 1$ and a non-linear map $f\colon V \to X$ that preserves semi-inner products. The map f may even be discontinuous unless X is strictly convex.

In this part of talk we manufacture an infinite-dimensional uniformly smooth Banach space $\mathfrak{X}$ and a nonlinear mapping $f\colon \mathfrak{X}\to\mathfrak{X}$ such that

$$[f(x)|f(y)] = [x|y], \quad x, y\in\mathfrak{X}.$$

Let $(Z, \|\cdot\|_o)$ be a two-dimensional normed space that is ***smooth*** but ***not strictly convex***. Then there are distinct vectors $u, w \in Z$ such that $\mathrm{conv}\{u, w\} \subseteq S_Z$. Without loss of generality, we may assume that $Z = \mathbb{K}^2$ as a vector space and $u = (-c, 1)$, $w = (c, 1)$ for some real number $0 < c < 1$. Thus $(0, 1) \in S_Z$. Moreover, without loss of generality we may assume that$(1, 0) \in S_Z$.

Lemma

Let $x \in \mathbb{K}$ and $\eta \in (0, c)$. Then $\|(\eta x, x)\|_o = \|(0, x)\|_o$.

Proof: Since $0 < \eta < c$, we have $\frac{\eta+c}{2c} \in [0, 1]$ and

$$(\eta, 1) = \left(1 - \tfrac{\eta+c}{2c}\right) u + \tfrac{\eta+c}{2c} w \in \mathrm{conv}\{u, w\} \subseteq S_Z.$$

Thus $(\eta, 1) \in S_Z$, *i.e.*, $\|(\eta, 1)\|_o = 1$. Since $(0, 1) \in S_Z$, $\|(0, 1)\|_o = 1$. Therefore $\|(\eta x, x)\|_o = |x| \cdot \|(\eta, 1)\|_o = |x| \cdot 1 = |x| \cdot \|(0, 1)\|_o = \|(0, x)\|_o$. ∎

We shall consider the space $\mathfrak{X} := \mathbb{K} \oplus_2 \ell_2(Z)$, the ℓ_2-sum of infinitely many copies of Z and the one-dimensional space. In other words,

$$\mathfrak{X} := \mathbb{K} \ \oplus_2 \ Z \ \oplus_2 \ Z \ \oplus_2 \ Z \ \oplus_2 \dots$$

and the norm in $\mathfrak{X}$ is thus given by

$$\|x\| := \sqrt{|x_1|^2 + \sum_{k=1}^{\infty} \|(x_{2k}, x_{2k+1})\|_o^2}, \tag{4}$$

where $x = (x_1, (x_2, x_3), (x_4, x_5), (x_6, x_7), \dots) \in \mathfrak{X}$.
The space $\mathfrak{X}$ is uniformly smooth because Z is smooth (hence uniformly smooth being finite-dimensional) and uniform smoothness passes to ℓ_2-sums of infinitely many copies of a uniformly smooth space [2, Corollary 4.9]. Since Z is isomorphic to the two-dimensional Hilbert space, X is isomorphic to $\ell_2(\ell_2^2)$, which is isometric to ℓ_2.

[2] T. Zachariades, *On ℓ_ψ spaces and infinite ψ-direct sums of Banach space*, Rocky Mt. J. Math., **41**(3): 971–997, 2011.

For a number $\eta \in (0, c)$, let $h_\eta \colon \mathfrak{X} \to \mathfrak{X}$ be a linear map given by

$$h_\eta(x_1, (x_2, x_3), (x_4, x_5), \dots) := (0, (\eta x_1, x_1), (x_2, x_3), (x_4, x_5), \dots). \quad (5)$$

Applying Lemma 1 to (4) we deduce that h_η is a linear isometry. Consequently,

$$[h_\eta(x)|h_\eta(y)] = [x|y] \quad (x, y \in \mathfrak{X},\ \eta \in (0, c)). \quad (6)$$

Combining (1) with (6) we may rearrange (6) as

$$\|h_\eta(y)\| \cdot \langle \varphi_{h_\eta(y)}, h_\eta(x) \rangle = [x|y] \quad (x, y \in \mathfrak{X},\ \eta \in (0, c)). \quad (7)$$

Moreover, putting y in place of x in (6) we get

$$\|h_\eta(y)\| = \|y\| \quad (\eta \in (0, c)). \quad (8)$$

We are now ready to construct the sought non-linear map that preserves semi-inner products.

For this, we fix a function $\gamma\colon [0,\infty) \to [0,\infty)$ with $\gamma(0) = 0$ such that $0 < \eta(x) < c$ $(x \in \mathfrak{X} \setminus \{0\})$ and γ is not constant on $(0,\infty)$. Next we choose a function $\eta\colon \mathfrak{X} \to [0,c)$ by

$$\eta(x) := \gamma(\|x\|) \quad (x \in \mathfrak{X}).$$

Then, we define a map $f\colon \mathfrak{X} \to \mathfrak{X}$ by the formula

$$f\left(x_1,(x_2,x_3),(x_4,x_5),\dots\right) := \left(0,\left(\eta(x)x_1,x_1\right),\left(x_2,x_3\right),\left(x_4,x_5\right),\dots\right).$$

Then we may recognise that

$$f(x) = h_{\eta(x)}(x) \quad (x \in \mathfrak{X}). \tag{9}$$

Consequently, f fails to be linear. However, in the case where

- γ is continuous and non-constant on $(0,\infty)$, f is continuous;
- γ is discontinuous on $(0,\infty)$, f is discontinuous too.

We *claim* that for all $x, y \in X$ we have $[f(x)|f(y)] = [x|y]$. For this, fix $x, y \in \mathfrak{X}$ and consider the associated maps $h_{\eta(y)}, h_{\eta(x)}\colon \mathfrak{X} \to \mathfrak{X}$.

Applying again Lemma 1 to (4) and (5), we conclude that

$$\|h_{\eta(y)}(y) + h_{\eta(x)}(y)\| = \|h_{\eta(y)}(y)\| + \|h_{\eta(x)}(y)\|.$$

It follows from the well-known property $\|a+b\|=\|a\|+\|b\| \Rightarrow \varphi_a = \varphi_b$ that $\varphi_{h_{\eta(y)}(y)} = \varphi_{h_{\eta(x)}(y)}$, *i.e.*,

$$\langle \varphi_{h_{\eta(y)}(y)}, w\rangle = \langle \varphi_{h_{\eta(x)}(y)}, w\rangle \quad (w \in \mathfrak{X}). \tag{10}$$

Consequently,

$$\begin{aligned}
[f(x)|f(y)] &\overset{(1)}{=} \|f(y)\|\cdot\langle \varphi_{f(y)}, f(x)\rangle \overset{(9)}{=} \|h_{\eta(y)}(y)\|\cdot\langle \varphi_{h_{\eta(y)}(y)}, h_{\eta(x)}(x)\rangle \\
&\overset{(10)}{=} \|h_{\eta(y)}(y)\|\cdot\langle \varphi_{h_{\eta(x)}(y)}, h_{\eta(x)}(x)\rangle \overset{(8)}{=} \|y\|\cdot\langle \varphi_{h_{\eta(x)}(y)}, h_{\eta(x)}(x)\rangle \\
&\overset{(8)}{=} \|h_{\eta(x)}(y)\|\cdot\langle \varphi_{h_{\eta(x)}(y)}, h_{\eta(x)}(x)\rangle \overset{(7)}{=} [x|y]\,.
\end{aligned}$$

This shows that $f\colon \mathfrak{X} \to \mathfrak{X}$ is indeed a non-linear map preserving semi-inner products.

Remark: In the above construction one may consider the ℓ_p-sums for $p \in (1, \infty)$ instead of the ℓ_2-sum. This will lead to a renorming of ℓ_p on which one may find a non-linear injection preserving the (unique) semi-inner products.

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Suppose that X is a Banach space and let $[\cdot|\diamond] : X \times X \to \mathbb{K}$ be fixed. Assume that $f : X \to X$ satisfy $[f(x)|f(y)] = [x|y], \quad x, y \in X$.

- Does ***strict convexity*** of X imply linearity of f?

If not, then the following open problem will seem to be natural:

- Suppose that X is ***uniformly strictly convex*** and ***uniformly smooth***. Are the mapping f linear?

If still not, then we will get another open problem:

- Characterize all functions $f : X \to X$ which satisfy$[f(x)|f(y)] = [x|y]$, $x, y \in X$.

Thank you for your attention.

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